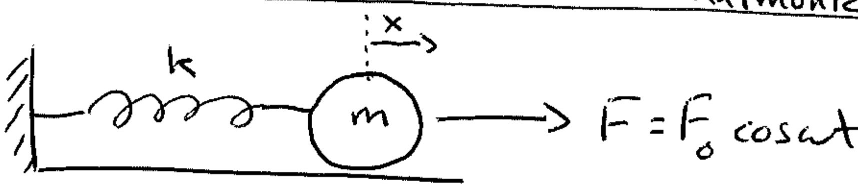
Classical & Quantum Waves

3) Forced oscillations

- There are many examples of oscillating systems that are driven periodically:
  - swing on playground
  - crystal-controlled clock (electrical pulses to maintain oscill.)
  - microwave oven driving oscillation of water molecules

3.1) Characteristics of forced harmonic motion

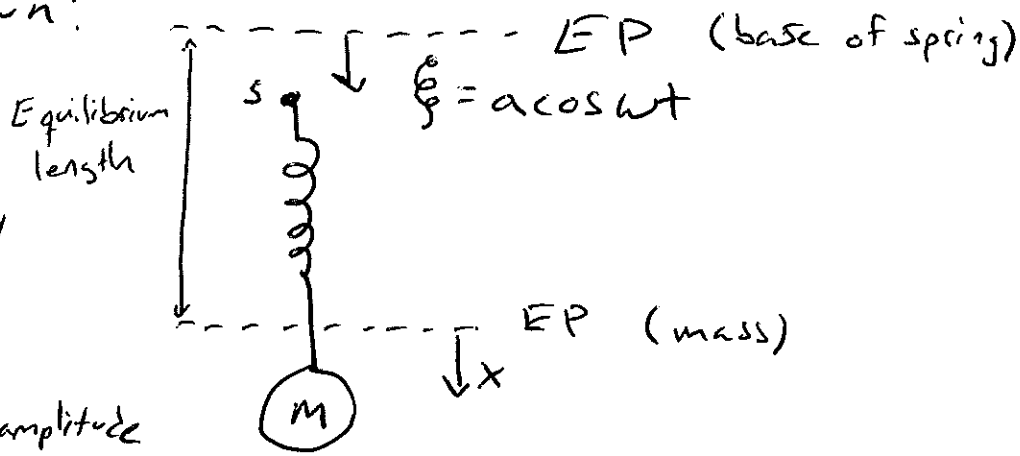


$$\boxed{m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t}$$

- When $F_0 = 0$, we recover the SHO

(ignore damping for now)

- Practical example: vertical mass/spring system, oscillate EP up and down:



- Low frequency drive
  - ↳ mass follows drive w/ same amplitude & phase
- ~Natural frequency:
  - ↳ Mass oscillates w/ max amplitude
- High frequency
  - ↳ oscillates w/ reduced amplitude and in opposite direction of drive
- very high frequency → amplitude vanishes!

* Systems <u>always</u> oscillate at the frequency of the applied driving Force!

- If frequency of drive ~ natural frequency of oscillator, get large amplitudes → <u>Resonance!</u>

3.2 <u>Eq. of motion for a forced harmonic oscillator (FHO)</u>

- When we oscillate the base of the spring up & down, we dynamically change the extension / compression of the spring. This creates forces that we need to consider

$$x \rightarrow x - \xi \qquad \left(\xi = \underset{\substack{\text{amp.}\\ \text{of osc.}\\ \text{suspension}}}{a} \; \cos \underset{\substack{\text{applied}\\ \text{frequency}}}{\omega} t \right)$$

Therefore:

$$m \frac{d^2 x}{dt^2} = -k(x - \xi)$$

$$\rightarrow \boxed{m \frac{d^2 x}{dt^2} + kx = ka \cos \omega t \equiv F_0 \cos \omega t} \qquad \text{w/ } F_0 = ka = m\omega_0^2 a$$

- We expect (from intuition / class demonstration) that the amplitude of oscillation strongly depends on drive frequency ω

Ansatz

$$\rightarrow \boxed{x = A(\omega) \cos(\omega t - \delta)}$$

phase angle b/t drive force and displacement

minus: displacement lags behind drive

- Compare with SHO: $x = A\cos(\omega t + \phi)$
  where ϕ, A determined by initial conditions

- Consider limit where drive frequency $\omega \to 0$

$$m\frac{d^2x}{dt^2} + kx = F_0 \underbrace{\cos\omega t}_{\to 1}$$

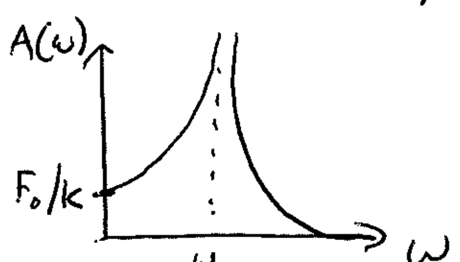
- since there is no more movement as $\omega \to 0$, the acceleration $\frac{d^2x}{dt^2} \to 0$

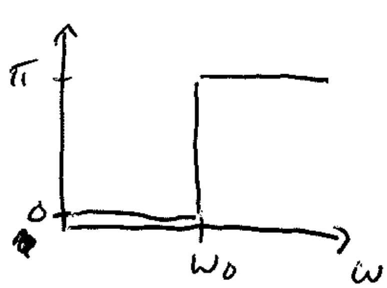
- This leaves us with $kx = F_0$, so the oscillation amplitude as $\omega \to 0$ approaches $x = \frac{F_0}{k}$

- since $F_0 = ka$, for $\omega \to 0$, $x \to a$

  ↳ meaning, the amplitude of the oscillating mass is the same as the amplitude a of the drive

- As ω is increased from 0, the amplitude of oscillation increases dramatically as resonance frequency is approached (ω_0)
- Past resonance ω_0, amplitude decreases toward 0





[Intuition up to now, now math]

- Going back to our ansatz that solution to diff. eq. is: $x = A(\omega)\cos(\omega t - \delta)$

- Plug into eq. $m\frac{d^2x}{dt^2} + kx = F_0 \cos\omega t$ w/ $F_0 = ka = m\omega_0^2 a$

$$\rightarrow -\omega^2 A(\omega)\underbrace{\cos(\omega t-\delta)}_{\text{expand these terms}} + \omega_0^2 A(\omega)\cos(\omega t - \delta) = \omega_0^2 a \cos\omega t$$

$$\rightarrow -\omega^2 A(\omega)(\cos\omega t\cos\delta + \sin\omega t \sin\delta) + \omega_0^2 A(\omega)(\cos\omega t \cos\delta + \sin\omega t \sin\delta)$$
$$= \omega_0^2 a \cos\omega t$$

- With some algebra, the above expression simplifies to:

(1st) $$A(\omega)\left(1 - \omega^2/\omega_0^2\right)\cos\delta = a$$

(2nd) $$A(\omega)\left(1 - \omega^2/\omega_0^2\right)\sin\delta = 0$$

(Here, we are equating coefficients of ~~$\cos\delta$~~ $\cos\omega t$ and ~~$\sin\delta$~~ $\sin\omega t$)

- Divide 2nd eq. by 1st, get $\frac{\sin\delta}{\cos\delta} = \tan\delta = 0$

↳ means that $\delta = 0$ or π (as expected)

- When $\delta = 0$, we have from 1^{st} eq.

$$A(\omega) = \frac{a}{(1 - \omega^2/\omega_0^2)} \qquad \text{(eq. 3.7)}$$

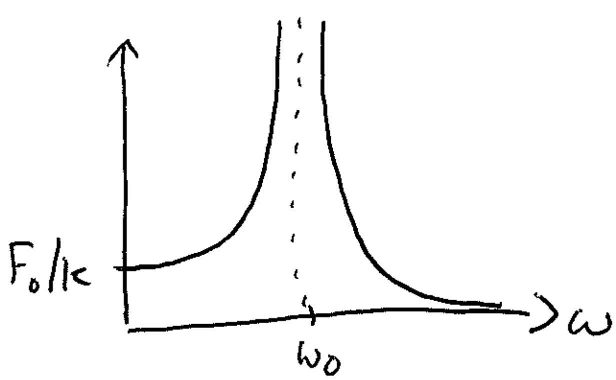
- Since we define $A(\omega)$ as a positive quantity, the above shows that $\omega < \omega_0$ when $\delta = 0$ (otherwise above would be negative)

- When $\delta = \pi$, we have ~~s~~ from 1^{st} eq.

$$\text{(eq. 3.8)} \quad A(\omega) = \frac{-a}{(1 - \omega^2/\omega_0^2)} \rightarrow \text{must be positive, so only true when } \omega > \omega_0$$

- Conclude that $x = A(\omega)\cos(\omega t - \delta)$ is a solution to our eq. of motion and that $\delta = 0$ for $\omega < \omega_0$, $\delta = \pi$ for $\omega > \omega_0$

- (eq. 3.7) shows that $A(\omega) \rightarrow a = \frac{F_0}{k}$ as $\omega \rightarrow 0$
- (eq. 3.8) shows that $A(\omega) \rightarrow 0$ as $\omega \rightarrow \infty$ ✓
- Also see that $A(\omega) \rightarrow \infty$ as $\omega \rightarrow \omega_0$



↳ This is clearly unphysical and only arises b/c we have neglected damping, which will make $A(\omega_0)$ finite in value